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"Comparing various fixed-point theorems and their applications across different fields of study".

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Abstract: In many branches of mathematics and the applied sciences, fixed-point theorems are essential resources. These theorems state that a function has at least one point, under particular assumptions, where the function value equals the point. The present study examines and contrasts various fixed-point theorems, namely Banach's Fixed-Point Theorem, Brouwer's Fixed-Point Theorem, and Kakutani's Fixed-Point Theorem, and their applications in many domains, including computer science, engineering, and economics. The comparison highlights these theorems' importance and adaptability in resolving challenging issues in a variety of academic fields.

Keywords: Fixed-Point Theorems, Banach's Fixed-Point Theorem, Brouwer's Fixed-Point Theorem, Kakutani's Fixed-Point Theorem.

Introduction: Fixed-point theorems provide a foundational concept in mathematical analysis, which states that a function will have at least one point that maps to itself under certain conditions. These theorems are not only of theoretical interest but also have practical applications in various fields. This paper aims to compare several key fixed-point theorems and discuss their relevance and applications in different disciplines. By understanding these theorems, we can better appreciate their role in solving real-world problems.Fixed-point theorems are important ideas in mathematics that help us find special points in functions where the output is equal to the input. These theorems are like powerful tools that guarantee the existence of these special points under certain conditions.

Stefan Banach's work in 1922 introduced one of the most famous fixed-point theorems, known as the Contraction Mapping Theorem or Banach Fixed-Point Theorem. It tells us that if we have a function that shrinks distances between points in a space, there must be at least one point in that space where the function value equals the point itself. This theorem is incredibly useful in fields like physics and engineering, where we use iterative methods to solve complex equations step by step.

Another important figure, Shizuo Kakutani, expanded on these ideas in 1941 with a generalization of Brouwer's Fixed-Point Theorem. Kakutani showed that even for more complicated functions that map to more than one point, there is always a fixed point. This extension is crucial in fields like economics and game theory, where it helps us understand stable situations where no one can improve their position without someone else losing out.

Review of Literature: The use of fixed-point theorems has been the subject of numerous investigations. The Contraction Mapping Theorem, sometimes referred to as Banach's Fixed-Point Theorem, is a frequently used tool in numerical analysis to demonstrate the uniqueness and existence of solutions to differential equations and iterative techniques. On the other hand, Brouwer's Fixed-Point Theorem is important in topology and has applications in economics and game theory. The Fixed-Point Theorem by Kakutani, which generalizes Brouwer's theorem, is crucial to understanding Nash equilibria in game theory. The basic research as well as current developments in the understanding of these theorems and their applications are reviewed in this section.

An extensive reference work on fixed-point theory is B.V. Rao's (4). It goes over the fundamental concepts, significant theorems, and applications of these concepts in real-world contexts like as the optimization of engineering and economic processes or the solution of differential equations.

The 2016 study by Sharma and Lal (5) investigates the use of fixed-point theorems in differential equations. They demonstrate how these theorems assist in demonstrating the existence and uniqueness of solutions to complex equations.

The application of fixed-point theorems in economics is covered in Mehta and Mookerjee's (6). They describe how the existence of stable economic conditions where demand and supply are balanced or where no one can profit more than everyone else is supported by these theorems.

Fixed-point theorems in partially ordered metric spaces were investigated in 2019 by Reddy and Ramesh (9). They investigated how these theorems can be used in scenarios where there is a hierarchy or order among the elements.

The study of Bhatt and Sarma (2020) was centered around fuzzy metric spaces and fixed-point theorems. In fuzzy metric spaces, the details of point-to-point distances are not necessarily present in a mathematical space. The application of fixed-point theorems in fractal geometry was investigated by Chatterjee and Majumdar in 2015 (11). Complex geometric forms known as fractals exhibit pattern repetition at all scales.

In 2017 (12), Joshi and Rana used the Banach's Fixed-Point Theorem to solve engineering challenges. They gave an example of how to apply this theorem to solve engineering designs and optimization problems. Their study brought to light how the theorem ensures efficiency and stability in a range of engineering applications.

In 2018, Das and Bose (13) investigated the game theory applications of Brouwer's Fixed-Point Theorem. They demonstrated how the theorem is necessary to demonstrate equilibrium point existence in economic models and strategic games. Their research demonstrated how the theorem affects competitive strategies and decision-making procedures.

Method: The techniques require a thorough analysis and comparison of the chosen fixed-point theorems. We examined each theorem's mathematical formulations, looked into its proofs and underlined the conditions in which it holds true. In addition, we examine how these theorems might be applied in other fields through the examination of case studies and real-world instances. We can comprehend each theorem's flexibility and applicability with this method.

Prior to delving into fixed-point theorems and their applications, we examined a number of significant scholarly works and publications that provide a basic understanding of these theorems. This required reading the theorems and comprehending their application to various branches of mathematics and science as well as their significance.
We then examined particular instances and case studies that employed these theorems. This made it easier for us to understand the real-world scenarios where identifying fixed points is essential, like in engineering to optimize designs or economics to comprehend stable prices.

We also examined the modifications made to these theorems to address various types of issues, such as fractal geometry, in which structures recur in intricate patterns, or fuzzy metric spaces, in which measurements are imprecise.
Through a comparative and analytical analysis of these various methods, we were able to identify the advantages and disadvantages of each theorem as well as their various realms of application. This clarifies how mathematics can be an effective tool for resolving real-life issues and understanding the world around us.

Banach's Fixed-Point Theorem

Theorem: Let $(\boldsymbol{X}, \boldsymbol{d})$ be a non-empty complete metric space and $\boldsymbol{T}: \boldsymbol{X} \to \boldsymbol{X}$be a contraction mapping (i.e., there exists a constant $\mathbf{0} \leq \boldsymbol{k} < 1$ such that $\boldsymbol{d}(\boldsymbol{T}(\boldsymbol{x}), \boldsymbol{T}(\boldsymbol{y})) \leq \boldsymbol{kd}(\boldsymbol{x}, \boldsymbol{y})$for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$). Then T has a unique fixed-point $\boldsymbol{x} * \in \boldsymbol{X}$.
Proof:Sequence Construction: Choose an arbitrary $\boldsymbol{x0} \in \boldsymbol{X}$ and define a sequence $\{\boldsymbol{x_n}\}$ by $\boldsymbol{x_{n+1}} = \boldsymbol{T}(\boldsymbol{x_n})$.
Contraction Property:
For $\boldsymbol{n} \geq \mathbf{1}$,

$$\boldsymbol{d}(\boldsymbol{x_{n+1}}, \boldsymbol{x_n}) = \boldsymbol{d}\big(\boldsymbol{T}(\boldsymbol{x_n}), \boldsymbol{T}(\boldsymbol{x_{n-1}})\big) \leq \boldsymbol{kd}(\boldsymbol{x_n}, \boldsymbol{x_{n-1}})$$

By induction, it followsthat
$d(x_{n+1}, x_n) \leq k^n \, d(x_1, x_0)$
Cauchy Sequence:
For $\boldsymbol{m} > n$,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n).$$

Using the contraction property,

$$d(x_m, x_n) \leq k^{m-1} d(x_1, x_0) + k^{m-2} d(x_1, x_0) + \cdots + k^n d(x_1, x_0)$$

This forms a geometric series,

$$\boldsymbol{d(x_m, x_n) \leq \frac{k^n d(x_1, x_0)}{1-k}}$$

Convergence:

Since $\mathbf{0} \leq \boldsymbol{k} < 1$, as $\boldsymbol{n} \to \infty, \boldsymbol{k^n} \to \mathbf{0}$. Thus, $\{\boldsymbol{xn}\}$ is a Cauchy sequence. Since $\boldsymbol{X}$ is complete, $\{\boldsymbol{xn}\}$ converges to some $\boldsymbol{x} * \in \boldsymbol{X}$.

Fixed Point:

$$x *= \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1})T(x *)$$

Uniqueness: Suppose $y *$ is another fixed point. Then,

$$d(x *, y *) = d(T(x *), T(y *)) \leq kd(x *, y *).$$

Since $k < 1$, the only solution is $d(x *, y *) = 0$, implying $x *= y *$.

Brouwer's Fixed-Point Theorem

Theorem: Every continuous function from a compact convex subset of a Euclidean space to itself has a fixed point.

Proof: Preliminaries: Let $\boldsymbol{K}$ be a compact convex subset of $\boldsymbol{R^n}$ and $\boldsymbol{f}: \boldsymbol{K} \to \boldsymbol{K}$ be continuous.

Sperner's Lemma:

Consider a simplicial decomposition of K. Color the vertices of the simplices according to a rule derived from f. By Sperner's Lemma, there exists a fully labeled simplex.

Approximation:

Approximate f. by a piecewise linear function g that maps vertices of the simplices to vertices according to f.. By Sperner's Lemma, g has a fixed point in one of the simplices.

Convergence:

As the simplicial decomposition gets finer, the fixed points of g converge to a fixed point of f.

Brouwer's Fixed-Point Theorem asserts that any continuous function mapping a compact convex set to itself has at least one fixed point. This theorem is fundamental in fields like economics, where it is used to prove the existence of equilibrium states.

Kakutani's Fixed-Point Theorem

Theorem: Let Xbe a non-empty compact convex subset of a Euclidean space, and let F:X→2 be an upper semi-continuous set-valued function with non-empty convex compact values. Then F has a fixed point.

Proof:Set-Valued Function:

F assigns to each $x \in X$a non-empty compact convex set $F(x) \subseteq X$.

Upper Semi-Continuity:

F is upper semi-continuous, meaning the inverse image of a closed set is closed.

Fixed Point Argument: Use the Fan-Glicksberg fixed-point theorem, which is an extension of the Schauder fixed-point theorem to set-valued maps, to show that F has a fixed point.

Application of Fan-Glicksberg:

The conditions of upper semi-continuity and compact convex values ensure the existence of a fixed point $x *\in F(x *)$.

Results: The comparison shows that although the main goal of all three theorems is to locate fixed points, there are differences in the fields in which they might be applied. Brouwer's theorem is essential to topology and economics, Banach's theorem is vital to iterative techniques and differential equations, and Kakutani's theorem is vital to game theory and economic models.

Banach's Fixed-Point Theorem provides a powerful tool for proving the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. This theorem is fundamental in various applications, especially in iterative methods for solving equations in mathematical analysis and applied fields

Conclusion: In mathematics and its applications in other fields, fixed-point theorems are essential. The fixed-point theorems by Banach, Brouwer, and Kakutani have been compared in this work, with special emphasis placed on their conditions, proofs and domains of application

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The Banach's theorem, which offers a framework for solving differential equations and other mathematical problems by repeated approximations, is essential to iterative approaches and numerical analysis. With its emphasis on continuous functions mapping compact convex sets, Brouwer's theorem is essential to topology and economics, especially for establishing the existence of equilibrium states. An expansion of Brouwer's theorem, Kakutani's theorem is crucial to game theory since it allows for the demonstration of Nash equilibria.

Banach's Fixed-Point Theorem guarantees that contraction mappings in complete metric spaces have a unique fixed point. The proof relies on constructing a sequence that converges to this fixed point, leveraging the contraction property to ensure the sequence is Cauchy and thus convergent in a complete space.

Future Directions: Extend the theorem to more general backgrounds, such as non-linear operators and mappings in different types of metric spaces and develop and refine iterative algorithms based on Banach's theorem for solving complex mathematical and engineering problems.

Explore the use of fixed-point theorems in the convergence analysis of algorithms in machine learning and optimization then Study the applications of Banach’s theorem in functional analysis, particularly in infinite-dimensional spaces and Banach spaces.

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